The Dependence Structure Between the Canadian Stock Market and the US/Canada Exchange Rate: A Copula Approach

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Abstract

In the present paper we investigate the dependence structure of real returns between the Canadian stock market and the US/Canada real exchange rate. We model dependence using the Symmetrized Joe-Clayton (SJC) copula with marginal models that depend on commodity and energy prices, the short term interest rate differential between Canada and the US, an AR(1) term and a GARCH process for the random error in returns. We estimate the SJC copula with montly data over the period 1995:1 to 2006:12. Our results show significant asymmetric tail dependence, such that the two returns are more dependent in the left than in the right tail of their joint distribution. Moreover, the assymetric tail dependence changes over time. We explain this asymmetry in terms of an asymmetric interest rate policy by Canadian monetary authorities in response to changes in commodity and energy prices

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1 Introduction

In recent times the interaction and dependence among money and asset markets, locally and internationally, have become stronger than in the past. There are many reasons for this development, including greater integration of financial markets, financial innovations, and the enormous flow of information among investors and policy makers, made possible by the internet and the existing information technologies. As a result, in the news media one can routinely observe overlaid indices of stock markets, exchange rates and interest rates, along with analyses of their co-movements and volatilities.

There are good reasons to examine formally the dependence structure of asset prices, including stock prices, interest rates and exchange rates. First, stock prices may affect aggregate demand through the wealth channel, and interest rates and exchange rates have a direct impact on investment demand and next exports, respectively. Thus, knowing the nature of their dependence may help policy makers and investment specialists **to** predict more accurately the position of aggregate demand and the macroeconomic equilibrium.

Second, a correct understanding of the dependence of asset process is important for proper risk measurement and portfolio diversification by financial analysts and private investors. Suppose for instance that the joint distribution of asset prices is asymmetric, such that there is higher probability of dependence in the left tail than in the right tail of the joint distribution. Then, if one assumes a symmetric joint distribution to assess and value risk, the assessment will be erroneous, as obviously downside and upside risks are different.

Third, modern central banks and investment houses have developed financial conditions indices (FCIs), as linear combinations of asset prices, in order to guide their policies and predict the state of the economy; see e.g., table 2 in Gauthier, Graham and Liu (2004).[1] Unless one has a good idea about the way in which the component parts of the FCIs are jointly dependent on each other, the predictive ability and practical usefulness of these indices will be limited.

Motivated by these considerations, in the present paper we examine the joint dependence structure between two important variables for the Canadian economy: the real stock market

[1] A well known index of this type is the monetary conditions index (MCI) developed by the Bank of Canada in the early 1990s and discontinued officially at the end of 2006. The MCI consisted of a linear combination of the short term nominal interest rate and the C-6 effective exchange rate, with weights 2/3 and 1/3 respectively, and it was intended as an indicator of Canada's monetary policy stance in relation to the official policy of inflation targeting. Clearly, given the MCI's definition, its behavior over time depended on the joint distribution of interest rates and exchange rates. Since the Canadian exchange rate is influenced not only interest rate changes but also by other factors, such as changes in commodity and energy prices, a careful study of the whole dependence structue between interest rates and exchange rates would be necessary in order for the MCI to serve as a useful guide of monetary policy stance. This dependence may have been one of the contributing factors for abandoning the MCI as an index of Canada's monetary policy stance at the end of 2006.

index, as proxied by the TSX index, and the US/Canada real exchange rate. The stock market index is potentially important through its wealth effects on consumption and the real exchange rate is an important variable, given the prominence of trade in the Canadian economy. Our analysis is based on copula functions, which have recently become an important alternative to modeling dependence in financial markets; see e.g., Chollete and Lu (2005), Forbes and Rigobon (2002), Hu(2005), Marchal and Zeevi(2002), Ning(2006), and Patton (2006). A copula is a function that connects the marginal cumulative distribution functions of random variables in order to recover their joint cumulative distribution (cdf). Further, by Sklar 's(1959) theorem, one can decompose the joint density function of a set of random variables into the product of the marginal cdf's and a copula function of the marginal cdf's that completely characterizes the dependence structure among the variables.[2]

In our analysis we first specify the marginal models for the two variables in terms of macroeconomic variables that are economically and statistically significant in explaining their time series properties over time. The common macroeconomic variables, in both marginal models, are commodity prices, energy prices and the interest rate differential between Canada and the US. Commodity and energy prices are used to capture the importance of the resources sector in the Canadian economy, and the interest rate differential is used to capture the monetary policy stance in Canada relative to the US. The marginal models are then completed by allowing for peresisrence in returns and a GARCH process for the error term in each model.

Since the copula is a function of the marginal distributions, its validity depends on the correct specification of the marginal models. For this reason, we perform several model misspecification tests in order to confirm the empirical adequacy of the marginal models. Next, we use the method of maximum likelihood and the Symmetrized Joe-Clayton (1997) (SJC) copula in order to estimate the dependence structure between the returns of the TSX index and the US/Canada exchange rate, conditional on the estimated marginal models. The SJC copula is chosen because, in addition to the other attractive properties of copula functions disscussed below, it allows for both symmetric and asymmetric tail dependence.

Figure 1 shows the joint histogram of the real US/Canada exchange rate returns r_{ex} and TSX stock returns r_{stock}, using monthly data over the period 1995:1 to 2006:12. As shown in the figure, there is evidence of asymmetric tail dependence between the two variables, with higher dependence in the left tail than in the right tail.

One objective of this paper is to test if this asymmetric tail dependence is statistically significant. Our test results indicate that this is indeed the case. Another objective is to explain the observed asymmetry in terms of economic and monetary policy considerations.

[2] The correlation coefficient is an inadequate measure of dependence unless the true joint distribution of asset prices is normal. First, correlation can measure only the strength of the linear association between random variables, but it cannot detect non-linear dependence, as may be the case in the tails of asymmetric distributions. Second, its value is dependent on outlier observations and may thus provide erroneous information of the underlying true relationship among random variables. Third, it is an average measure of linear dependence over the whole range of the random variables and thus it provides, at best, information about the degree of dependence but not the structure of dependence among them. Knowledge of the latter is crucial if one wishes to study extreme or tail dependence, as is the case in **the present paper.**

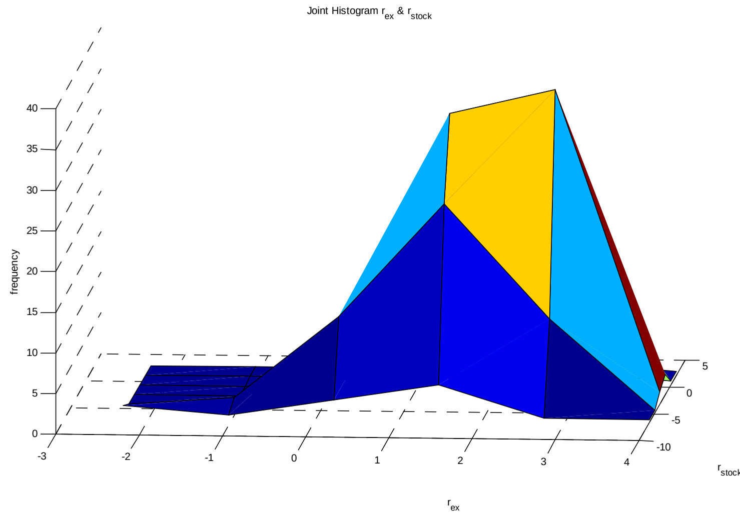

Figure 1: Joint Histogram of r_{ex} and r_{stock}

First, we expect changes in commodity and energy prices to have a symmetric effect on the exchange rate and TSX returns: when commodity and energy prices increase both returns will tend to increase. The TSX returns will increase, because the TSX index is dominated by resource and energy companies, and the Canadian dollar will gain in value against the US dollar because higher commodity and energy prices mean greater demand for the Canadian dollar in foreign exchange markets. The opposite will be the case when these prices decline in international markets. On the other hand, changes in the short term interest rates are likely to affect **more** exchange rate returns than the stock returns, as the latter depend more on long-term interest rates and capital investments by Canadian firms.

Second, given their explicit inflation targeting policy that keeps inflation under control, monetary authorities in Canada may act asymmetrically in response to movements in commodity and energy prices. For instance, when commodity and energy prices fall in international markets, the monetary authorities may have no incentive to increase interest rates relative to the US in order to support the Canadian dollar, as this action would induce a domestic currency appreciation. In contrast, a weaker Canadian dollar gives a competitive advantage to the domestic export sector and prevents unemployment from rising at home. As a result the US/Canada exchange rate returns may slide further down the left tail in its joint distribution with the TSX returns. On the other hand, when commodity and energy prices rise in international markets, the monetary authorities have an incentive to reduce interest rates relative to the US in order to counteract the ensuing appreciation of the Canadian dollar and avoid a further erosion of Canada's competitive advantage in international markets. Consequently the Canadian dollar will not appreciate as much and then the right tail of the joint distribution of the two variables will be rather short.

Clearly, if the Canadian monetary authorities respond in this asymmetric mode, using

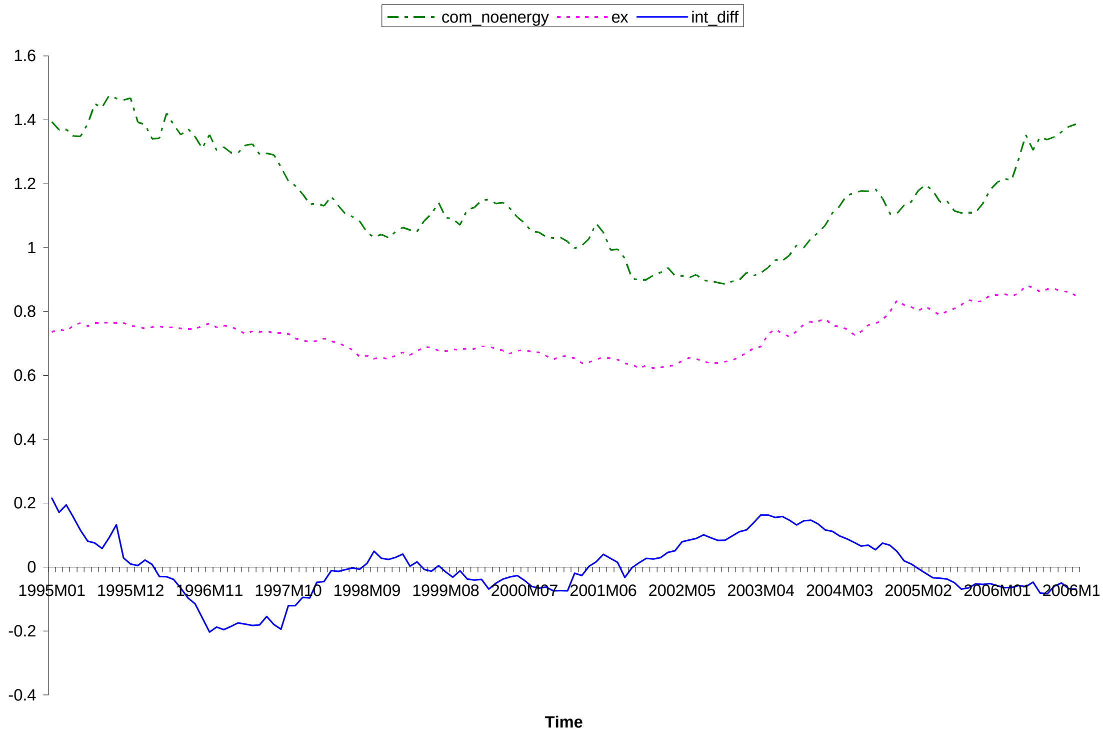

Figure 2: Commondity Prices, the Real Exchange Rate and the Interest Rate Differential

short-term interest rates, their actions would have an impact on the joint distribution of the stock and the exchange rate returns that would be detectable in the actual data. There is some evidence in the data in support of such behavior by the monetary authorities. Figure 2 plots the real commodity prices, the real exchange rate and the short term interest rate differential between Canada (net of inflation targeting) and the US from 1995:1 to 2006:12. Overall the interest rate differential is low or decreasing which is consistent with the policy asymmetry argument that we make. Up to the early 1990s when commodity prices were falling and the Canadian dollar was depreciating the interest rate differential was either falling or remained flat. Since 2003, when commodity prices started increasing continuously, the interest rate differential has followed a downward trend and became negative since early 2005.

The rest of the paper is organized as follows. Section 2 discusses briefly copula functions and their main properties, and outlines the estimation method and related measures of dependence. Section 3 describes the data. Section 4 discusses the marginal models, the joint model based on the SJC copula, and discusses the empirical results. Section 5 concludes the paper.

2 Copula Functions, Estimation and Measures of Dependence

2.1 Copula Functions

A copula function is a multivariate cumulative distribution function of which the marginal distribution is uniform on the interval [0,1]. A copula is useful because it can be used to analyze the dependence structure of variables in a multivariate distribution. This is justified by a fundamental result known as Sklar's theorem. For simplicity, we consider the bivariate case.

Sklar's Theorem: Let $F_{XY}(.)$ be a joint distribution function with margins $F_X(.)$ and $F_Y(.)$. Then there exists a copula C(.) such that for all x, y in R,

$$F_{XY}(X, Y; \theta_x, \theta_{y,}\theta_c) = C(F_X(x; \theta_x), F_Y(y; \theta_y), \theta_c). \tag{1}$$

If $F_X(.)$ and $F_Y(.)$ are continuous, then $C(.)$ is unique; otherwise, $C(.)$ is uniquely determined on Range$F_X\times$RangeF_Y. Conversely, if $C(.)$ is a copula and $F_X(.)$ and $F_Y(.)$ are the marginal cumulative distribution functions, then the function $F_{XY}(.)$ defined by (1) is a joint cumulative distribution function with margins $F_X(.)$ and $F_Y(.)$.

By Sklar's theorem, a joint distribution can be decomposed into its univariate marginal distributions, and a copula, which captures the dependence structure between the variables X and Y. As a result, copulas allow us to model the marginal distributions and the dependence structure of a multivariate random variable separately.

Different copulas usually represent different dependence structures with the so called association parameters θ_c indicating the strength of the dependence. Some commonly used copulas in economics and finance include: the bivariate Gaussian copula, the student-T copula, the Gumbel copula, the Clayton copula, and the Symmetrized Joe-Clayton (SJC) copula. We discuss the SJC copula in detail in section 3 and provide a brief description of the other copulas in the Appendix.

Copulas have several desirable properties. One of the key properties of copulas is that they are invariant under increasing and continuous transformations. This property is useful, as **such** transformations are commonly used in economics and finance. For example, the copula is invariant to logarithmic transformation of variables. This is not true for the correlation coefficient, which is invariant only under linear transformations. Another property of copulas is that they provide complete information about the structure of dependence among random variables over their whole range of joint variation and not just over certain portions of it. A third useful property of copulas, that we exploit in this study, is that it provides analytic measures of dependence in the tails of its joint distribution.

2.2 Estimation

Equation (1) leads naturally to a maximum likelihood estimation. To see this first differentiate both sides of equation (1), to get

$$f_{XY}(x,y,\theta_x,\theta_{y,}\theta_c)=f_X(x,\theta_x)\cdot f_Y(y,\theta_y)\cdot c(u,v,\theta_x,\theta_{y,}\theta_c), \tag{2}$$

where, after suppressing the parameters for convenience, $f_{XY}(x,y)$, $f_X(x)$, $f_Y(y)$ and $c(u,v)$ are density functions given by

$$\begin{aligned} f_{XY}(x,y) &= \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y},\ f_X(x)=\frac{\partial F_X(x)}{\partial x},\ f_Y(y)=\frac{\partial F_Y(y)}{\partial y}, \\ c(u,v) &= \frac{\partial^2 C(F_X(x),F_Y(y))}{\partial u \partial v}, \end{aligned}$$

with $u=F_X(x)$, $v=F_Y(y)$.

Next, log-linearizing equation (2), we get

$$L_{XY}(\theta_x,\theta_{y,}\theta_c)=L_X(\theta_x)+L_Y(\theta_y)+L_C(\theta_x,\theta_{y,}\theta_c), \tag{3}$$

where $L_{XY}=\log(f_{XY}(x,y))$, $L_X=\log(f_X(x))$, $L_Y=\log(f_Y(y))$, $L_C=\log(c(u,v))$. The one step full maximum likelihood (ML) estimator of the parameters in L_{XY} is obtained simply by maximizing L_{XY} with respect to these parameters. Under standard regularity conditions the ML estimator is consistent, asymptotically efficient, and asymptotically normal.

Joe and Xu (1996) proposed an alternative two-step procedure to estimate the parameters $(\theta_x,\theta_{y,}\theta_c)$. In the first step, one estimates the marginal parameters θ_x and θ_y by maximizing L_X and L_Y, respectively. In the second step, one estimates the copula parameters θ_c by maximizing L_C, given the estimated parameters for the marginal models. Joe (1997) proved that, under proper regularity conditions, the two-step estimator is also consistent and asymptotically normal. This procedure is easy to implement and convenient when there are many parameters to be estimated. Since our models involve a large number of parameters, we adopt this two-step estimation method in the present study.

2.3 Related Measures of Dependence

The well known Spearman's ρ and Kendall's τ rank correlation coefficients provide alternative nonparametric measures of dependence between variables that, unlike the simple correlation coefficient, do not require a linear relationship between the variables. For this reason they are commonly studied with copula models. The relationship between Spearman's ρ and copulas is as follows:

$$\rho=12\int_0^1\int_0^1 uv dC(u,v)-3 \tag{4}$$

Thus ρ depends on marginal distributions. There is also an exact relationship between copulas and Kendall's τ, which, for variables X and Y, is defined as the difference between the probability of the concordance and the probability of the discordance: The higher the τ value, the stronger the dependence between X and Y. The relationship between Kendall's τ and copulas is as follows:

$$\tau = 4\int_0^1 \int_0^1 C(u,v)dC(u,v) - 1 \tag{5}$$

Thus, Kendall's τ depends on the copula function and not on the marginal distributions.

Another useful dependence measure based on copulas is tail dependence, which is used to measure co-movements of variables in extreme situations. Tail dependence measures the probability that both variables are in their lower or upper joint tails. Intuitively, upper (lower) tail dependence refers to the relative amount of mass in the upper (lower) quantile of their joint distribution. Because tail dependence measures are derived from copula functions, they possess all the desirable properties of copulas mentioned above. The lower (left) and upper (right) tail dependence coefficients are defined as

$$\lambda_l = \lim_{u \longrightarrow 0} Pr[F_Y(y) \leq u | F_X(x) \leq u] = \lim_{u \longrightarrow 0} \frac{C(u,u)}{u}, \tag{6}$$

$$\lambda r = \lim_{u \longrightarrow 1} Pr[F_Y(y) \geq u | F_X(x) \geq u] = \lim_{u \longrightarrow 1} \frac{1 - 2u + C(u,u)}{1-u}, \tag{7}$$

where λ_l and $\lambda r \in [0,1]$. If λ_l or λr are positive, X and Y are said to be left (lower) or right (upper) tail dependent; see Joe (1997) and Nelson (1999).

3 Data

For the empirical analysis we constructed monthly series from January of 1995 to December of 2006 for the following variables: US/Canada real exchange rate, real TSX index, real commodity and energy prices, and short term Canadian and US interest rates. We chose the starting point January 1995 since there is a structural break in the data in mid-1990s associated with Canada becoming an important net energy exporter; Issa et. al. (2006).

The nominal US/Canada exchange rate is defined as US dollars per Canadian dollar. The exchange rate, the TSX index, and the commodity price and energy price indices were obtained from CANSIM-II of Statistics Canada. These variables were then expressed in real US dollars, and measured in logs. The US/Canada real exchange rate R_t was constructed by multiplying the nominal exchange rate by the ratio of the Canadian CPI to the US CPI. The CPI indices were obtained from the IMF statistics through the DataStream website. The real TSX index S_t was obtained by dividing the product of the nominal TSX index and the nominal exchange rate by the US CPI index. The commodity and energy price indices were deflated by the US CPI index in order to construct their real counterparts $P_{com,t}$ and $P_{en,t}$, respectively. The Canadian and US interest rates, $I_{ca,t}$ and $I_{us,t}$, are the three month treasury bill rates of the two countries and the data were obtained from DataStream. Since Canada has followed an official inflation targeting policy over the sample period, we projected the

Canadian interest rates off the gap between actual and target inflation of 2%, by regressing the Canadian treasury bill rates on this gap and using the residual in the estimation of the models below. Using these interest rates we constructed the Canada-US interest rate differential $I_t = I_{ca,t} - I_{us,t}$. Finally, the returns for the exchange rate and the TSX index $r_{ex,t}$ and $r_{stock,t}$ were calculated as 100 times the differences of the real exchange rate and the real TSX index, respectively, between the month t and the month t-1.

4 Model Specification, Tests and Results

The theory of copulas outlined above applies **to** stationary time series. However, as shown in Table 1, all the variables in the present study, including the TSX index and the real exchange rate, have unit roots, and are thus integrated of order one. For this reason, we examine the dependence structure between the log differences, (or rates of return) of the US/Canada real exchange rate and the real TSX index, which are stationary. Further, as shown in the last two rows of Table 1, the real exchange rate and the real TSX index are cointegrated with commodity and energy prices, and thus we specify the marginal models for the two variables in error correction form. In what follows, we specify the two marginal models for the real exchange rate and TSX returns, and the joint model represented by the SJC copula.

Table 1: Unit root test for stationary and co-integration

Null Hypotheses: unit root of series below	p-values from ADF test	**Reject** null at 10%
R_t	0.73[12]	No
S_t	0.79[12]	No
$P_{com,t}$	0.42[12]	No
$P_{en,t}$	0.76[12]	No
$I_{ca,t}$	0.21[12]	No
$I_{us,t}$	0.40[12]	No
$r_{ex,t}$	0.00 [4]	Yes
$r_{stock,t}$	0.00[4]	Yes
I_t	0.01[12]	Yes
residual from regress R_t on $\{P_{com,t}, P_{en,t}\}$	0.02[4]	Yes
residual from regress S_t on $\{P_{com,t}, P_{en,t}\}$	0.08[1]	Yes

Note: Note that numbers in brackets are lags used in the Augmented Dickey Fuller test.

4.1 The Marginal Models

4.1.1 Specification of marginal models

The marginal models consist of two error correction models (ECMs), each extended to allow for an autoregressive AR(1) component and a GARCH(1,1) process for the error term. The lag length in these terms was determined by significance testing based on the data; (section 4.1.2 below). Hence, we specify the marginal models as follows:

$$r_{ex,t} = \alpha_0(R_{t-1} - \beta_0 - \beta_c P_{com,t-1} - \beta_e P_{en,t-1}) + \alpha_1 I_{t-1} + \alpha_2 r_{ex,t-1} + \varepsilon_{ex,t}, \quad (8)$$

$$\sigma^2_{ex,t} = \theta_0 + \theta_1 \sigma^2_{ex,t-1} + \theta_2 \varepsilon^2_{ex,t-1}, \sqrt{\frac{1}{\sigma^2_{ex,t}}} * \varepsilon_{ex,t} \sim \text{iid N}(0,1), \quad (9)$$

$$r_{stock,t} = \delta_0 + \delta_1(S_{t-1} - \gamma_0 - \gamma_c P_{com,t-1} - \gamma_e P_{en,t-1}) + \delta_2 I_{t-1} + \delta_3 r_{stock,,t-1} + \varepsilon_{stock,t}, \quad (10)$$

$$\sigma^2_{stock,t} = \varphi_0 + \varphi_1 \sigma^2_{stock,t-1} + \varphi_2 \varepsilon^2_{stock,t-1}, \sqrt{\frac{\upsilon}{\sigma^2_{stock,t} * (\upsilon - 2)}} * \varepsilon_{stock,t} \sim \text{iid t}_{\upsilon}, \quad (11)$$

where R_{t-1} and S_{t-1} are the real exchange rate and the real TSX index at time $t-1$, $P_{com,t-1}$ and $P_{en,t-1}$ are the lagged real commodity and real energy prices, I_{t-1} is the lagged Canada-US short-term interest rate differential, $\sigma^2_{ex,t}$ and $\sigma^2_{stock,t}$ are the variances of $\varepsilon_{ex,t}$ and $\varepsilon_{stock,t}$ and $N(0,1)$ and t_{υ} are the standard normal distribution and the t-distribution with υ of degrees of freedom, respectively.

The ECM component of the marginal model for the real exchange rate is due to Amano and van Norden (1995). This equation incorporates a long-run relationship between the real exchange rate and commodity and energy prices, in recognition of the fact that resource sectors play an important role in the Canadian economy. It also allows for short-run fluctuations in the real exchange rate caused by changes in the short-term Canada-US interest rate differential.

Since Canada has a sizable natural resources sector and the TSX index includes many resource and energy companies, we consider a similar marginal model for the TSX stock returns as well. Our empirical results in section 4.1.2 below provide significant evidence in support of this specification of Canadian stock returns.

Finally, as shown in Table 2, both the Jarque-Bera and Lilliefors normality tests reject normality of the monthly TSX returns $r_{stock,t}$, at the 5% level of significance. As a result, we specify a t-distribution for the standardized error term of the TSX returns. The student t-distribution is used to capture fat-tails or leptokurtosis in stock returns which is a common feature in many financial time series. We determine the degrees of freedom ν of the t-distribution by the estimation of the marginal model in Section 5. The Jarque-Bera normality test rejects the normality of the returns $r_{ex,t}$ for the exchange rate, but the Lilliefors normality test does not reject normality. Since the Jarque-Bera test is an asymptotic test while the Lilliefors test is more suitable for finite samples, we specify a normal distribution for $r_{ex,t}$ based on the Lilliefors normality test results.

4.1.2 Results for the Marginal Models

Now, we consider the results for the marginal models specified in equations (8) to (11). The results are shown in Table 3. We added autoregressive terms in each marginal model in order to capture possible persistence in stock and real exchange returns. The lag length in

Table 2: Normality Test of Returns

	Type of Tests	$r_{ex,t}$	$r_{stock,t}$
Jarque Bera test	Test statistic	9.54 (0.017)	171.14 (0.001)
	Reject normality	Yes	Yes
Lilliefors test	Test statistic	0.047 (0.500)	0.09 (0.006)
	Reject normality	No	Yes

Note: The numbers in brackets are p-values from the test.

the AR1 component in each model was determined by specifying the maximum lag length at 12 and deleting the insignificant autoregressive terms, using a 5% level of significance. As shown in Table 3 persistence is significant in stock returns up to the first order and marginally significant in the exchange rate returns. Also, as shown in Table 3, there is evidence the error terms in equations (8) and (10) are heteroskedastic, and that heteroskedasticity in the data is captured adequately by the generalized autoregressive conditional heteroskedasticity GARCH(1,1) model of Bollerslev (1987). As a result we include a GARCH(1,1) process for each return, given by equations (9) and (11).

Table 3: Marginal Models

Variables	$r_{ex,t}$	$r_{stock,t}$
Intercept	-40.28 (-88.29)**	385.87 (194.17)**
$P_{com,t-1}$	0.45 (19.77)**	0.24 (2.41)**
$P_{en,t-1}$	0.14 (17.84)**	0.50 (14.24)**
Adj. R_Square	0.83	0.59
Intercept (in ECM)	-	1.15 (2.43)**
Speed of adjustment	-0.037 (-0.95)	-0.06 (-1.93)
I_{t-1}	3.57 (2.26)**	-3.03 (-0.60)
AR1	0.15 (1.58)	0.21 (2.32)**
ARCH0	0.11 (0.57)	2.24 (0.71)
ARCH1	0.05 (0.8)	0.09 (0.93)
GARCH1	0.89 (5.56)**	0.81 (4.25)**
TDF_inverse	-	0.16 (2.58)**
Total R_Square	0.09	0.07

Note: numbers in brackets are t values. ** indicate significant at 5% level.

For the US/Canada exchange rate model (column 2 of Table 3), an increase of the commodity price or the energy price leads to an increase in US/Canada exchange rate, implying an appreciation of the Canadian dollar. This is consistent with the view that Canada is a commodity and energy dependent economy, and a net exporter of energy products; Issa et. al. (2006). The coefficient of the lagged interest rate differential is significantly positive. Thus an increase of Canada's short term interest rate relative to the US interest rate increases the US/Canada exchange rate, thereby strengthening the Canadian dollar. This is expected, as the increase of the interest rate causes a greater demand for the Canadian dollar. The exchange rate returns are also positively affected by their own lagged values, as is shown by its significant positive value, which indicates persistence of the exchange rate returns. The GARCH term is strongly significant, indicating conditional heteroskedasticity of the exchange rate returns.

The results for the TSX marginal model are presented in the last column of the Table 3. Both the energy prices and the commodity prices have significant positive effects on the TSX returns. This is consistent with the fact that Canada is a resource and energy based economy. Again the AR1 term and the GARCH term are statistically significant, implying persistence and conditional heteroskedasticity of the stock returns. The degrees of freedom TDF of the t-distribution is 6.25 (1/0.16) and are statistically significant, confirming that the stock returns are not normal. The coefficient of the interest rate differential is negative but not statistically significant. Thus, in contrast to the exchange rate returns, the Canada-U.S interest rate differential does not have a significant effect on the TSX returns. This result is not surprising, as changes in the short term interest rates are more likely to affect money markets than stock markets which depend principally on long term rates of return of capital investments by firms.

Overall, the results from the marginal models indicate that in the long run, both the Canada/US exchange rate and the TSX index are driven by commodity and energy prices. In the short run, the interest rate differential has an asymmetric effect on the change of the Canada/US exchange rate and the TSX index: **a** positive effect on exchange rate returns and no effect on stock returns.

As noted earlier, the joint copula model requires the correct specification of the marginal models. If the marginal distributions are not correct, their probability transforms will not be iid uniform(0,1), and hence the copula model will be misspecified. We employ two goodness-of-fit tests for our marginal models: Lagrange Multiplier (LM) tests for serial independence of the probability transforms, and Kolmogorov-Simirnov (K-S) tests to test if the probability transforms are uniform(0,1). Specifically, the LM tests are applied to the first four moments of u_{ex} and v_{stock}, defined as the cumulative probability transforms of the the standardized residuals from the marginal models of the two returns. For this, we regress $(u_{ex,t} - \overline{u}_{ex})^k$ and $(v_{stock,t} - \overline{v}_{stock})^k$ on 10 lags of each variable, for k=1,2,3,4. The test statistic $(T-20)R^2$ from each regression follows an asymptotic χ^2_{20} distribution under the null hypothesis.

Table 4 reports the results for the LM and K-S tests. The p-values from the LM tests range from 6% to 96%, implying that we cannot reject the null hypothesis that the marginal cdfs are serially independent. The p-values from the K-S tests are 91% for the exchange rate margin and 67% for the stock index margin, indicating the hypothesis that each of the probability transforms of the two marginal distributions is uniform(0,1) is not rejected. Thus both marginal models pass the LM and K-S tests at the 5% level of significance. These results provide significant evidence that our marginal models are correctly specified. Hence our copula model can correctly capture the dependence structure of the two returns.

Table 4: Test of the Marginal Distribution Models

Test	r_{ex}	r_{stock}
First moment LM test	0.24	0.96
Second moment LM test	0.15	0.94
Third moment LM test	0.10	0.81
Fourth moment LM test	0.06	0.61
K-S test	0.91	0.67

Note: The table provides p-values from LM tests of serial independence and the p-value from the Kolmogorov-Smirnov (K-S) tests for the adequacy of the distribution model.

4.2 The Joint Model and the SJC Copula

As shown in Figure 1, there is empirical evidence of asymmetric tail dependence in the Canadian stock reruns and the real exchange rate returns. Based on this evidence, we select a flexible copula function to model the joint distribution of the two returns.

4.2.1 The SJC copula model

The SJC copula allows for both asymmetric upper and lower tail dependence and symmetric dependence as a special case. For this reason, we choose the SJC copula for the joint model.[3]

The SJC copula is a modification of the so called "BB7" copula of Joe (1997). It is defined as

$$C_{SJC}(u,v|\lambda r,\lambda_l) = 0.5 \times (C_{JC}(u,v|\lambda r,\lambda_l) + C_{JC}(1-u,1-v|\lambda_l,\lambda r) + u + v - 1), \quad (12)$$

where $C_{JC}(u,v|\lambda r,\lambda_l)$ is the BB7 copula (also called Joe-Clayton copula) defined as

$$\begin{aligned} & C_{JC}(u,v|\lambda r,\lambda_l) \\ = \; & 1-(1-\left\{\left[1-(1-u)^k\right]^{-r}+\left[1-(1-v)^k\right]^{-r}-1\right\}^{-1/r})^{1/k}, \end{aligned} \quad (13)$$

where $k = 1/log_2(2-\lambda r)$, $r = -1/log_2(\lambda_l)$, and $\lambda_l \in (0,1)$, $\lambda r \in (0,1)$.

By construction, the SJC copula is symmetric when λ_l=λr. The density of the SJC copula is

$$\begin{aligned} c_{SJC}(u,v|\lambda r,\lambda_l) &= \frac{\partial^2 C_{SJC}(u,v|\lambda r,\lambda_l)}{\partial u \partial v}, \\ &= 0.5 \times [\frac{\partial^2 C_{JC}(u,v|\lambda r,\lambda_l)}{\partial u \partial v} + \frac{\partial^2 C_{JC}(1-u,1-v|\lambda_l,\lambda r)}{\partial(1-u)\partial(1-v)}], \end{aligned} \quad (14)$$

[3] The Gaussian and T copulas are used frequently to model dependence in economics and finance. However, they are not the most suitable alternatives in our case of potentially asymmetric tail dependence. **As shown in the Appendix, the** Gaussian copula forces zero tail dependence and the T copula imposes symmetric tail dependence. Moreover, the T copula requires that the degrees of freedom for the margins are the same. As we discuss in section 4 below, this constraint is not satisfied with our data. **The Gumbel copula can capture positive right tail dependence but no left tail dependence. And the Clayton copula allows for positive left tail dependence and zero right tail dependence.**

where

$$
\begin{aligned}
c_{JC}(u,v|\lambda r,\lambda_l) &= \frac{\partial^2 C_{JC}(u,v|\lambda r,\lambda_l)}{\partial u \partial v} \\
&= (AB)^{-r-1}(1-u)^{k-1}(1-v)^{k-1} \\
&\quad \times\{[1-(A^{-r}+B^{-r}-1)^{-1/r}]^{-1+1/k}(A^{-r}+B^{-r}-1)^{-2-1/r}(1+r)k \\
&\quad +[1-(A^{-r}+B^{-r}-1)^{-1/r}]^{-2+1/k}(A^{-r}+B^{-r}-1)^{-2-2/r}(k-1)\} \quad (15)
\end{aligned}
$$

$k = 1/log_2(2-\lambda r)$, $r = -1/log_2(\lambda_l)$, $A = 1-(1-u)^k$, and $B = 1-(1-v)^k$.

The functional form of $\frac{\partial^2 C_{JC}(1-u,1-v|\lambda_l,\lambda r)}{\partial(1-u)\partial(1-v)}$ is the same as that of $\frac{\partial^2 C_{JC}(u,v|\lambda r,\lambda_l)}{\partial u \partial v}$. If we substitute u and v for $1-u$ and $1-v$ in the latter we get the former. Also $k = 1/log_2(2-\lambda_l)$ and $r = -1/log_2(\lambda r)$ for the former.

The copula log-likelihood function is $L_C = log(c_{SJC}(u,v|\lambda r,\lambda_l))$. Thus, the **static** tail dependence parameters λr and λ_l can be easily estimated by maximizing L_C. In the next section we estimate λr and λ_l twice: one time when the variables u, v in the SJC copula are the empirical distribution functions of the stock market and real exchange rate returns, and another time when the variables u, v are the cumulative distribution functions of the standardized residuals from the marginal models. We interpret similarity of the estimates of λr and λ_l across the two cases as additional evidence in favour of the empirical adequacy of our two marginal models in equations (8) to (11).

We also examine the possibility of dynamic or time varying tail dependence in the data. In particular, following Patton (2006), we estimate the following ARMA-type process for tail dependence:

$$
\begin{aligned}
\lambda_{l,t} &= (1+\exp(-h_{l,t}))^{-1}, \quad \lambda_{r,t} = (1+\exp(-h_{r,t}))^{-1}, && (16) \\
h_{l,t} &= h_{l,0} + \beta^l h_{l,t-1} + \gamma_l \sum_{j=1}^{p} |u_{t-j} - v_{t-j}|, && (17) \\
h_{r,t} &= h_{r,0} + \beta^r h_{r,t-1} + \gamma_r \sum_{j=1}^{p} |u_{t-j} - v_{t-j}|. && (18)
\end{aligned}
$$

This model contains an autoregressive term designed to capture persistence in dependence, and a variable which is a mean absolute difference between u and v. The latter variable is positive when the two probability integral transforms are on the opposite side of the extremes of the joint distribution and close to zero when they are on the same side of the extremes. The logistic transformation of the ARMA process guarantees that the tail dependence parameters lie in the [0,1] interval.

4.2.2 Results for the Dependence Structure

Now we consider the results of dependence between the two returns. As a first step, we examine three conventional measures of dependence: the simple linear correlation coefficient, Spearman's rank correlation and Kendall's rank correlation. As shown in Table 5 the linear correlation coefficient between the stock returns and the exchange rate returns is 0.50, indicating that an increase (decrease) of the TSX index is associated with the appreciation (deprecation) of the Canadian dollar. Spearman's rho is 0.49, indicating strong rank correlation. Kendall's tau coefficient is 0.34, showing the probability of concordance is significantly higher than the probability of discordance.

Table 5: Correlation Coefficients

	$r_{ex,t}$ $r_{stock,t}$
Leanear correlation	0.50
Spearman rank correlation	0.49
Kendall rank correlation	0.34
Number of Observations	144

Next, in order to get a sense of the dependence structure in the data, following Knight, Lizieri and Satchel (2005), we calculate an empirical copula table. To do this, we first rank the pairs of return series in ascending order and then we divide each series evenly into 6 bins. Bin 1 includes the observations with the lowest values and bin 6 includes observations with the highest values. We want to know how the values of one series are associated with the values of the other series, especially whether lower returns in the TSX index are associated with lower returns in the US/Canada exchange rate. Thus, we count the numbers of observations that are in cell (i, j). The dependence information we can obtain from the frequency table is as follows: if the two series are perfectly positively correlated, most observations lie on the diagonal; if they are independent, then we would expect that the numbers in each cell are about the same; if the series are perfectly negatively correlated, most observations should lie on the diagonal connecting the upper-right corner and the lower- left corner; if there is positive lower tail dependence between the two series, we would expect that more observations in cell (1,1). If positive upper tail dependence exists, we would expect large number in cell (6,6).

Table 6: Joint Frequency Table

Frequency	1	2	3	4	5	6	Total
1	11	2	7	1	1	2	24
2	7	8	2	5	2	0	24
3	2	7	3	4	4	4	24
4	1	4	4	5	6	4	24
5	1	0	3	6	7	7	24
6	2	3	5	3	4	7	24
Total	24	24	24	24	24	24	144

Table 6 shows the dependence structure. Cell (1,1) has a joint frequency of 11, which means that out of 144 observations, there are 11 occurrences when both the TSX index and the US/Canada exchange rate returns lie in their respective lowest 6th percentiles (1/6th quantile). This number is the largest among all cells, and it is much bigger than numbers in other cells, pointing to evidence of lower tail dependence. There are 7 occurrences in cell (6,6), which is not apparently larger than other cells, indicating no or not strong evidence of upper tail dependence. Clearly, the table shows evidence of asymmetric tail dependence.

Next we estimate and test for asymmetric tail dependence between the two returns, using the SJC copula twice. First we examine tail dependence by applying the copula model to the unconditional returns of the two series. Since the returns are conditionally heteroskedastic, we filter the returns with a GARCH(1,1) model and use the empirical CDF from each series to estimate the copula model. The results are shown in the first row of Table 7. There exists significant lower tail dependence but insignificant upper tail dependence, with the lower tail dependence being about 78% higher than the upper tail dependence. This suggests that the probability of the Canadian dollar depreciating heavily against the US dollar, given that the TSX index drops deeply, is about 78% higher than the probability of the corresponding appreciation given an extremely big increase of the TSX index. This implies that the U.S/Canada exchange rate and the TSX index are more dependent during extreme downturns than during extreme upturns of the two markets.

Second, we estimate the copula model using the CDFs of the standardized residuals from the two marginal models. The results are shown in the second row of Table 7. As in the unconditional case, there exists only significant lower tail dependence but insignificant upper tail dependence. Further, over 74% (0.29/0.40) of the left tail dependence in the unconditional copula model is explained by the conditional copula model. This again implies that our marginal models are well specified

Table 7: Tail Dependence

	λ_{low}	λ_{Upper}
r_{ex} and r_{stock}	0.3964(4.7698)**	0.2221(1.7185)
From the model	0.2895(2.8632)**	0.2069(1.5300)

Note: ** indicates significant at 5% level

Table 8 reports the results for dynamic tail dependence from estimating equations (16)-(18). For lower tail dependence, the intercept and the persistence parameters are statistically significant at the 5% and 10% level respectively, implying significant time varying lower tail dependence. On the other hand, in the case of upper tail dependence, none of the ARMA parameters is statistically significant. This is consistent with the findings in the static case, in which upper tail dependence is not significant at the 5% level. Figure 3 shows the time variation in the two tail dependence measures. First, the lower tail dependence measure is changing over time and is more volatile than the upper tail dependence measure. Second, the lower tail dependence plot lies above the upper tail dependence plot most of the time, implying asymmetric of the tail dependence over time. In addition, the mean of the dynamic

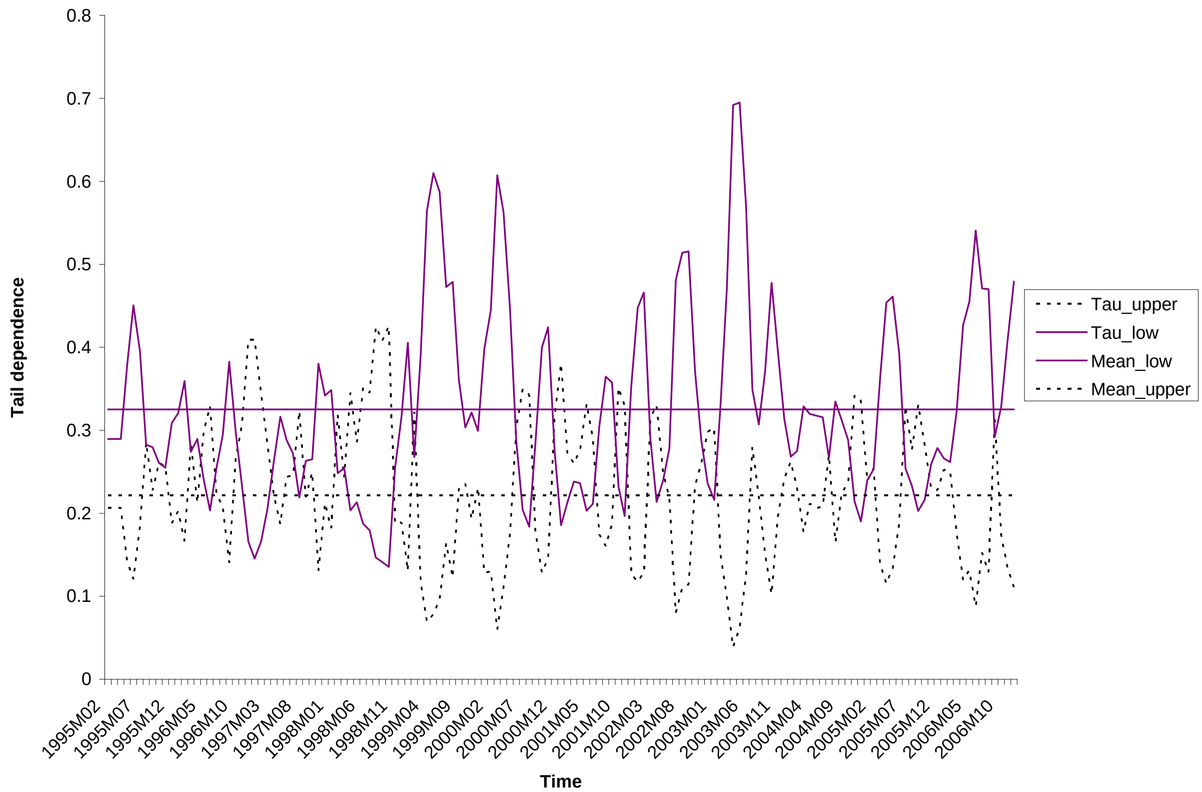

Figure 3: Dynamic Tail Dependence

lower dependence lies high above the mean of the dynamic upper tail dependence, indicating that, on average, the lower tail dependence is much greater than upper tail dependence in scale.

Table 8: Dynamic Tail Dependence

λ_l			λ_r		
-2.1912	3.9160	1.4850	0.2076	-5.7149	-0.8119
(-2.6582)**	(1.5974)*	(0.8060)	(0.0932)	(-0.8366)	(-0.1549)

Note: Numbers in the brackets are t values. ** indicates significant at 5% level

Overall, the empirical results point to asymmetric static and dynamic tail dependence between the real exchange rate and real TSX returns, with lower tail dependence significantly different from zero and upper tail dependence not significantly different from zero. Our conditional copula model, based on the marginal models, goes a long way in accounting for the observed asymmetric tail dependence.

We can throw more light on this asymmetry in terms of the results of the estimated marginal models and a plausible behavior by Canadian monetary authorities. As shown in Table 3, both the real exchange rate and real TSX returns respond positively and significantly to commodity and energy price increases, leading to a positive linear correlation between the two series. However, the effect of the short term interest rate differential is positive and significant for the real exchange returns and negative but insignificant for the TSX returns. Consequently, when the commodity and energy prices fall in international markets, the Canadian dollar depreciates and the TSX index drops, leading to lower tail dependence. The opposite is the case when commodity and energy prices increase. It is worth noting that the estimated total effect of commodity and energy price changes is smaller for exchange rate returns than TSX returns (0.59 vs 0.74). As a result exchange rate returns decrease less than TSX returns when commodity and energy prices fall.

How can we then explain the greater lower tail than upper tail dependence between the two returns? As we have conjectured in the introduction, when inflation is under control, the Canadian monetary authorities may have no incentive to increase the short term interest rate and support the Canadian dollar when commodity and energy prices fall, as a low value of the Canadian dollar is good for Canadian exports and employment. Given the estimated interest rate differential effect, this policy stance will tend to move the exchange rate returns further down in the left tail of the joint distribution, thereby causing a greater lower tail dependence between the two variables than would be the case otherwise. On the other hand, when the commodity and energy prices rise in international markets, both the Canadian dollar and the TSX index will rise, leading to the upper tail dependence between the two returns. However, too high commodity and energy prices and the appreciation of the Canadian dollar reduce exports and employment. In this case, the Bank of Canada has an incentive to reduce interest rates and contain the appreciation of the Canadian dollar, thereby reducing the upper tail dependence. This asymmetric monetary policy stance combined with the results of the estimated marginal models can then **provide a plausible explanation** of the dependence structure between the real US/Canada exchange rate and real TSX returns that we noted in Figure 1 and formally detected in our copula estimations and hypothesis testing.

5 Conclusions

In this paper, we examined the extreme co-movements between the US/Canada real exchange rate and the real TSX returns using the copula approach. This method of studying dependence among economic time series is useful because it can be used to study not only the degree of dependence among random variables, such as asset prices, but also their structure of dependence, including asymmetric dependence in the tails of their joint distribution.

The marginal models that we used in the empirical analysis were motivated by the statistical properties of the variables in the analysis and the underlying structure of the Canadian economy. Both models were thus cast in error correction form, similar to that for the real exchange rate in the existing literature. To account for persistence and heteroskedasticity in the two series, the marginal models were extended to include autoregressive and GARCH terms.

We studied the dependence in the rates of returns of the TSX index and the US/Canada exchange rate using the symmetrized Joe Clayton (SJC) copula for the joint model. This is a flexible copula function because it allows for asymmetric tail dependence as well as symmetric dependence as a special case.

Our empirical results point to significant static and dynamic asymmetric tail dependence between the two rates of return, with the lower tail dependence being significantly greater than upper tail dependence. This implies that Canada's main stock market index and real exchange rate vs the US dollar are more dependent in global downturns than global upturns. In the present study, we explained this evidence in terms of the empirical results of our marginal models, and a specific behavior by Canadian monetary authorities that act asymmetrically, through interest policies, in response to movements in commodity and energy prices that cause fluctuation in the real exchange rate.

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Appendix

Commonly used copulas:

Bivariate Gaussian copula

The bivariate Gaussian copula is defined as

$$C(u, v; \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v), \rho), \tag{19}$$

where $0 \leq u, v \leq 1$ and $-1 \leq \rho \leq 1$. Φ_ρ is the bivariate normal distribution function with correlation coefficient ρ, and Φ^{-1} is the inverse of the univariate normal distribution function. By Sklar's theorem, we can have

$$H(x, y) = C(F(x), G(y)) = \Phi_\rho(\Phi^{-1}(F(x)), \Phi^{-1}(G(y)), \rho). \tag{20}$$

That is we can construct bivariate distributions with non-normal marginal distributions and the Gaussian copula.

The relationship between Kendall's tau and ρ for Gaussian copula is:

$$tau = \frac{2}{\pi} \arcsin(\rho), \tag{21}$$

Gaussian copula has zero tail dependence, therefore $\lambda_l = \lambda r = 0$.

T copula

The T copula is defined as

$$C_{\upsilon,\rho}(u, v) = t_{\upsilon,\rho}(t_\upsilon^{-1}(u), t_\upsilon^{-1}(v)),$$

where $t_{\upsilon,\rho}$ is the bivariate student t distribution with degree of freedom υ and the correlation coefficient ρ. t_υ^{-1} is the inverse of the univariate student t distribution.

Its Kendall's tau can be expressed as a function of ρ $tau = \frac{2}{\pi} \arcsin(\rho)$.

The T copula has symmetric tail dependence with dependence coefficient as follows

$$\lambda_l = \lambda r = 2t_{\upsilon+1}(-(\upsilon+1)^{1/2}(1-\rho)^{1/2}(1+\rho)^{-1/2}).$$

Gumbel copula

The Gumbel copula is defined as

$$C_\alpha(u, v) = \exp(-((-\ln u)^\alpha + (-\ln v)^\alpha)^{\frac{1}{\alpha}}), \text{ for } \alpha \in (0, 1],$$

where a is the associate parameter.

The Kendall's τ and the associate parameter is linked by the following equation: $\alpha = 1/(1-\tau)$.

The Gumbel copula has no left tail dependence but positive right tail dependence. The tail dependence coefficients can be written as $\lambda_l = 0$, and $\lambda r = 2 - 2^{1/\alpha}$.

Clayton copula

The Clayton copula is defined as

$$C_\alpha(u,v) = (u^{-\alpha} + v^{-\alpha} - 1)^{1/\alpha} \text{ for } \alpha > 0$$

where α is the associate parameter.

The associate parameter can be expressed as a function of Kendall's tau as $\alpha = 2\tau/(1-\tau)$.The Clayton copula does not have right tail dependence but has left tail dependence as $\lambda_l = 2^{-1/\alpha}$ and $\lambda r = 0$.